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A Simple Proof of the Continuity of Expected Payoffs

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Abstract. In games with compact pure strategy spaces, the continuity of the payoff functions is preserved in the expected payoff functions of the mixed extension. This note provides an elementary proof of this fact, showing that the commonly assumed Hausdorff property is, in fact, not needed.

Keywords. Compact games, expected payoffs, weak* topology, continuity, Hausdorff separability axiom.

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1 Introduction

For n -player games with compact Hausdorff pure strategy spaces and continuous payoff functions, [Glicksberg \(1952\)](#) asserted that a player's expected payoff is continuous with respect to the product of the weak* topologies on the corresponding mixed strategy spaces. As the original work did not include a detailed proof, a small but significant literature has since developed, offering complete arguments under varying assumptions and employing a range of techniques. In the general case, [Glycopantis and Muir \(2000\)](#) applied the Stone–Weierstrass Theorem, while [Zarichnyi \(2004\)](#) relied on Milyutin maps. Focusing on metrizable spaces, [Aliprantis et al. \(2006\)](#) used tools from functional analysis, such as the Closed Graph Theorem.¹

In this note, we present an alternative proof that avoids advanced techniques, relying instead on an elementary characterization of continuity on a product space. The argument also shows that the commonly imposed Hausdorff assumption is unnecessary.

Our auxiliary characterization of continuity is as follows: A real-valued function $f(x, y)$ defined on a product of compact spaces is continuous if and only if it is continuous in x , uniformly over y , and separately continuous in y , for each fixed x . Since uniform continuity in x with respect to y is preserved under taking expectations, this characterization provides exactly the right tool to lift continuity properties from a game to its mixed extension.

The remainder of this note is organized as follows. Section 2 presents the auxiliary characterization of joint continuity. Section 3 applies this result to establish the continuity of expected payoffs in the mixed extension.

¹For related results, see also [Becker and Damianov \(2006\)](#), [Kozhan and Zarichnyi \(2008\)](#), and [Kim \(2014\)](#).

2 A Characterization of Joint Continuity

The proof of our main result is based on the following characterization of the continuity of a function defined on the product of two topological spaces.

Lemma 1. *Let X and Y be topological spaces, and let $f : X \times Y \rightarrow \mathbb{R}$. Suppose that f is continuous and that Y is compact. Then:*

- (i) *f is continuous in x , uniformly over $y \in Y$; and*
- (ii) *f is continuous in y , for every $x \in X$.*

Conversely, if conditions (i) and (ii) hold, then f is jointly continuous on $X \times Y$.

Proof.² Let x_α be a net converging to $x_0 \in X$, and let $\varepsilon > 0$. For any $y_0 \in Y$, by continuity, we find an index α_0 and an open neighborhood $V \equiv V(y_0)$ of y_0 such that $|f(x_\alpha, y) - f(x_0, y_0)| < \frac{\varepsilon}{2}$, for any $\alpha \geq \alpha_0$ and $y \in V$. By shrinking V , if necessary, we also have $|f(x_0, y) - f(x_0, y_0)| < \frac{\varepsilon}{2}$, for any $y \in V$. Hence, $|f(x_\alpha, y) - f(x_0, y)| < \varepsilon$, for any $\alpha \geq \alpha_0$ and $y \in V$. By compactness, we find $y_1, \dots, y_K \in Y$ such that $Y = \bigcup_{k=1}^K V(y_k)$. Moreover, for each $k \in \{1, \dots, K\}$, there exists α_k such that $|f(x_\alpha, y) - f(x_0, y)| < \varepsilon$, for any $\alpha \geq \alpha_k$ and $y \in V(y_k)$. Therefore, $|f(x_\alpha, y) - f(x_0, y)| < \varepsilon$, for any $\alpha \geq \max\{\alpha_1, \dots, \alpha_K\}$ and $y \in Y$. This yields condition (i). As condition (ii) is obvious, this proves the first part.

Conversely, let (x_α, y_α) be a net converging to $(x_0, y_0) \in X \times Y$, and let $\varepsilon > 0$. By condition (i), there exists α_1 such that $|f(x_\alpha, y) - f(x_0, y)| < \frac{\varepsilon}{2}$, for any $\alpha \geq \alpha_1$ and $y \in Y$. In particular, $|f(x_\alpha, y_\alpha) - f(x_0, y_\alpha)| < \frac{\varepsilon}{2}$, for any $\alpha \geq \alpha_1$. By condition (ii), there exists α_2 such that $|f(x_0, y_\alpha) - f(x_0, y_0)| < \frac{\varepsilon}{2}$ for any $\alpha \geq \alpha_2$. Hence, $|f(x_\alpha, y_\alpha) - f(x_0, y_0)| < \varepsilon$, for any $\alpha \geq \max\{\alpha_1, \alpha_2\}$. This proves the second part, and hence the lemma. $\square$

²The use of nets in this proof was kindly suggested by the Associate Editor. The first part of Lemma 1 corresponds to [Cohn \(2013, Lem. 7.6.3\)](#). The proof of that part is, consequently, added for completeness only.

3 The Continuity of Expected Payoffs

Let $G = (S_i, u_i)_{i=1}^n$ be an n -person game, where S_i is the set of player i 's pure strategies and $u_i : S_1 \times \dots \times S_n \rightarrow \mathbb{R}$ is player i 's payoff function, for $i \in \{1, \dots, n\}$. It is assumed that each S_i is compact, but not necessarily Hausdorff, and that each u_i is jointly continuous. Let $\mathcal{P}(S_i)$ denote the space of all regular probability measures on the Borel subsets of S_i , with typical element μ_i . In the *mixed extension* of G , each player $i \in \{1, \dots, n\}$ chooses some $\mu_i \in \mathcal{P}(S_i)$. [Glicksberg \(1952\)](#) defined player i 's expected payoffs as the iterated expectation

$$E_{\mu_1, \dots, \mu_n}[u_i] = \int_{S_1} \left\{ \dots \left\{ \int_{S_n} u_i(s_1, \dots, s_n) d\mu_n(s_n) \right\} \dots \right\} d\mu_1(s_1).$$

We may use Lemma 1 to check that the iterated integral is well-defined. From continuity, $u_i(s_1, \dots, s_n)$ is continuous in $(s_1, \dots, s_{n-1})$, uniformly over s_n . Hence, the innermost integral $\int_{S_n} u_i(s_1, \dots, s_n) d\mu_n(s_n)$ is continuous in $(s_1, \dots, s_{n-1})$. Straightforward induction over n yields the claim.³

Let $\mathcal{C}(S_i)$ denote the space of continuous (and therefore bounded) functions on S_i . We define the *weak* topology* on $\mathcal{P}(S_i)$ as the coarsest topology such that the evaluation map $e_f : \mu_i \mapsto \int_{S_i} f(s_i) d\mu_i(s_i)$ is continuous, for any $f \in \mathcal{C}(S_i)$. [Glicksberg \(1952\)](#) introduced the weak* topology by embedding $\mathcal{P}(S_i)$ into the adjoint of the Banach space $\mathcal{C}(S_i)$, i.e., via the Riesz Representation Theorem. Clearly, in the Hausdorff case, our definition of the weak* topology coincides with Glicksberg's definition.

The following result establishes the continuity of expected payoffs in games with compact pure strategy spaces.

³Since the set of Dirac measures is weak* dense in the space of Borel probability measures, this argument extends to show that $E_{\mu_1, \dots, \mu_n}[u_i]$ is independent of the order of integration. For details on this point, see the working paper version ([Ewerhart, 2025](#)).

Theorem 1. *Suppose that $S_1, \dots, S_n$ are compact (but not necessarily Hausdorff), and that u_i is continuous. Then, the mapping $(\mu_1, \dots, \mu_n) \mapsto E_{\mu_1, \dots, \mu_n}[u_i]$ is continuous with respect to the product of the weak* topologies.*

Proof. By Lemma 1, $u_i(s_1, \dots, s_n)$ is continuous in $(s_1, \dots, s_{n-1})$, uniformly over s_n . Hence,

$$E_{\mu_n}[u_i] = \int_{S_n} u_i(s_1, \dots, s_n) d\mu_n(s_n)$$

is continuous in $(s_1, \dots, s_{n-1})$, uniformly over μ_n . Further, by the definition of the weak* topology, $E_{\mu_n}[u_i]$ is continuous in μ_n , for any fixed $(s_1, \dots, s_{n-1})$. Therefore, using Lemma 1 again, $E_{\mu_n}[u_i]$ is continuous in $(s_1, \dots, s_{n-1}, \mu_n)$. Proceeding by induction, $E_{\mu_1, \dots, \mu_n}[u_i] = E_{\mu_1}[\dots [E_{\mu_n}[u_i]] \dots]$ is seen to be continuous in $(\mu_1, \dots, \mu_n)$, as has been claimed. $\square$

Conflict of Interest Statement

There are no conflicts of interest to declare.

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